Stationary spectrum of vorticity cascade in two-dimensional turbulence

Claudia Pasquero
Istituto di Scienze dell'Atmosfera e del Clima, CNR, Corso Fiume 4, 10133 Torino, Italy

Gregory Falkovich
Department of Physics of Complex Systems, Weizmann Institute, 76100 Rehovot, Israel

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The logarithmic renormalization predicted by Kraichnan (1971) for the direct cascade of enstrophy in the inertial range of two-dimensional turbulence has been observed in a numerical simulation. A moderate resolution allows for a very long time integration that provides very good statistics. Deviations from Gaussianity in the vorticity probability distribution are observed.

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Turbulence of incompressible fluid resists theoretical description because of nonlinearity and nonlocality. The only rigorous arguments are the flux relations for the conserved quantities. For example, two-dimensional (2D) Navier-Stokes equation can be written for the vorticity $\omega=\nabla\times\mathbf{v}$ as

$$\frac{\partial\omega}{\partial t}+(\mathbf{v}\cdot\nabla)\omega=f+d, \tag{1}$$

where f and d are forcing and dissipative terms, respectively. For the distances r_{12} that are smaller than the scale of the forcing L yet larger than the dissipation scale, the flux for the vorticity cascade is as follows:

$$\langle(\mathbf{v}_1-\mathbf{v}_2)\cdot\nabla\omega_1\omega_2\rangle\simeq\text{const.} \tag{2}$$

The rest of the correlation functions are, strictly speaking, unknown. To achieve some theoretical progress, Batchelor, Kraichnan, and Leith suggested to exploit the analogy between vorticity and passive scalar. The scalar satisfies the same equation $\partial\theta/\partial t+(\mathbf{v}\cdot\nabla)\theta=f+d$ with the only difference that $\mathbf{v}$ is independent of θ [1–3]. In the absence of pumping and damping, vorticity and passive scalar are both Lagrangian invariants, i.e., they are transported unchanged by the fluid flow. This allows one to argue, for instance, that the pair correlation function $\langle\theta_1\theta_2\rangle$ is proportional to the average time needed by two fluid particles to separate from r_{12} to L [4,5]. Power counting in Eq. (2) suggests the scaling $v_1-v_2\propto r_{12}$. If the velocity field is spatially smooth the fluid particles separate exponentially, giving $\langle\theta_1\theta_2\rangle\propto\lambda^{-1}\ln(L/r_{12})$, where λ is the Lyapunov exponent of the flow. Had vorticity behaved exactly as a passive scalar, the pair correlator $\langle\omega_1\omega_2\rangle\propto\ln(L/r_{12})$ would have given the energy spectrum $E(k)=k|\mathbf{v}_k|^2\propto k^{-3}$ [1–3]. Note that the logarithmic correlator of vorticity means that velocity is not exactly smooth but has at least logarithmic singularities, contradicting the initial assumption $v_1-v_2\propto r_{12}$. Clearly, vorticity is not passive and the separation of fluid particles is sensitive to the value of vorticity they carry (for example, separation is suppressed inside vortices, i.e., regions of strong vorticity). Generally, analogies between active and passive quantities may be very misleading, as they are, for instance, between vorticity and magnetic field in 3D and velocity and passive scalar in the Burgers equation [4]. However, the analogy is very useful in our case since the main difference between vorticity and passive scalar is the renormalization of the Lyapunov exponent $\lambda(r)$. For vorticity cascade, the Lyapunov exponent depends on the scale r. Indeed, the speed of particle separation is determined by the fluid motions at scales larger than r; as the distance grows there are less and less such motions and the speed of separation decreases. This is opposite to what takes place in a nonsmooth velocity field where the separation growth is due to smaller scale motions, leading to Richardson's superdiffusion. Assuming the velocity gradients to be δ correlated in time, Kraichnan found the law of renormalization $\lambda(r)\propto\ln^{1/3}(L/r)$, which gives the spectrum [6]

$$E(k)\propto k^{-3}\ln^{-1/3}(kL). \tag{3}$$

The same result can be derived by different uncontrollable closures based on weak phase coherence [7]. Even though velocity gradients are long correlated in the Lagrangian frame (their correlation time is larger than the inverse Lyapunov exponent), the same renormalization holds [5]. Let us stress that the arguments of [5,6] are plausible self-consistency checks at best and cannot be considered as rigorous proofs. Here we show that the logarithmic renormalization can be observed in numerical simulations.

The logarithmic factor is difficult to detect in numerical simulations, where statistical fluctuations introduce some noise in the energy spectrum. For this reason, the inertial range behavior of two-dimensional turbulence is still in debate [8–10]. Moreover, even though one can show that energy spectra $E(k)\propto k^{-a}$ with $a>3$ are structurally unstable so that an ultimate small-scale asymptotics has to have $a=3$ [4,5], coherent structures may steepen the energy spectrum at intermediate wave numbers [11,12]. Freely decaying turbulence spontaneously develops coherent vortices [13] and is not the appropriate case for studying Kraichnan scaling. A particular forcing that does not allow the formation of coherent structures has to be introduced. To this purpose, we use a forcing δ correlated in time. A constant energy flux is injected at large scales $(4\leqslant k\leqslant 5)$. The phases of the corresponding Fourier modes are randomly changed at each integration step. The Fourier transform of the force is as follows: $\hat{f}(k)=F(k)e^{i\theta_k}$. Here $F(k)=F_0$ for $4\leqslant k\leqslant 5$ and zero else-

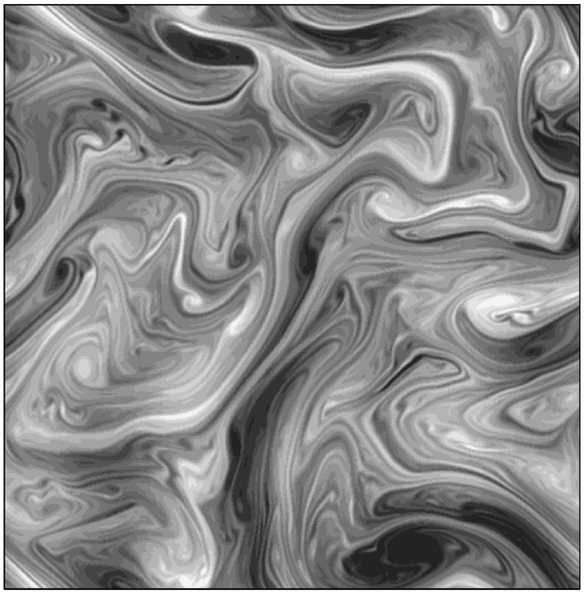

FIG. 1. A snapshot of the vorticity field in the statistically stationary regime. Here, $\nu = 6\times10^{-17}$ and $\mu = 0.12$. The numerical integration has been performed with a pseudospectral method in a doubly periodic square domain of size $[2\pi, 2\pi]$, with (512×512) grid points. All the variables and parameters are dimensionless.

where, and θ_k is a random variable with a uniform distribution between 0 and 2π. The dissipative term $d = -\mu\omega + \nu\nabla^8\omega$ accounts for two different kinds of dissipation: the first term represents linear Ekman friction, while the second one represents small-scale viscous dissipation. Instead of the standard molecular dissipation proportional to $\nabla^2\omega$, here we consider an iterated Laplacian ($\nabla^8\omega$) to confine dissipation to small scales, such that the Kolmogorov scale, k_ν, is close to the smallest resolved scale. Such hyperviscosity is of common use in numerical simulations, even though one cannot be totally certain that it does not affect the properties of the spectrum in the inertial range. With an appropriate choice of parameter values, this system does not develop coherent structures [14,15].

Starting from any initial condition, force and dissipation eventually compensate each other, leading to a statistically steady configuration where mean energy $E = \langle(\nabla\psi)^2\rangle/2$ and mean enstrophy $Z = \langle(\nabla^2\psi)^2\rangle/2$ are conserved. Angular brackets here indicate an average over the whole domain. After the statistically stationary state is reached, the integration is continued for a very long time (more than a thousand eddy turnover times $T_Z = Z^{-1/2}$). A snapshot of the vorticity field in this regime is shown in Fig. 1. During this period of time, the variations around the mean values are within 1% for the energy and within 4% for the enstrophy. From the vorticity equation (1), the equation for the enstrophy spectrum $Z(k)$ can be obtained

$$\frac{1}{2}\frac{\partial Z(k)}{\partial t} + \frac{\partial \Phi(k)}{\partial k} = \mathcal{F}(k) - \mu Z(k) - \nu k^8 Z(k). \qquad (4)$$

Here $\Phi(k)$ is the enstrophy flux, and $\mathcal{F}(k)$ is the enstrophy forcing. In the stationary regime, the time derivative vanishes. Force is confined to a limited wave-number band, therefore, at the other wave numbers the nonlinear term $\partial\Phi(k)/\partial k$ has to be balanced by the dissipative one. We have verified that below the forcing scale ($kL>1$) the enstrophy flux $\Phi(k)$ is at least an order of magnitude larger than the friction dissipation. In the range $k_F<k<k_\nu$ (where $k_F = 1/L$) forcing and dissipation are negligible or zero, and the enstrophy flux is nearly constant, $\Phi(k) = \eta$. The energy spectrum (3) was suggested precisely for this inertial range. The time averaged (dimensionless) energy spectrum calculated in our simulation is shown in Fig. 2(a). Figure 2(b) shows the compensated spectra, where the compensation function is $\eta^{-2/3}k^3$ (dotted line), and $\eta^{-2/3}k^3[\ln(k/k_F)]^{1/3}$ (solid line). We believe that the logarithmic correction that is observed is significant. Indeed, a power function steeper than k^{-3} does not provide a fit as good as the logarithmic corrected function does. Such an accurate fit is made possible here by very large statistics that smooth out the statistical noise in the spectrum. We believe that our data together with an alternative approach by spectral reduction [8] support logarithmic renormalization of the energy spectrum in the vorticity cascade.

Other studies have been recently performed on the direct cascade, showing steep energy spectra, and providing theoretical arguments for the existence of power-law spectra, whose scaling depends on the friction coefficient [16,17]. However, the inertial range is absent in those cases because the large friction does not allow for the existence of a range

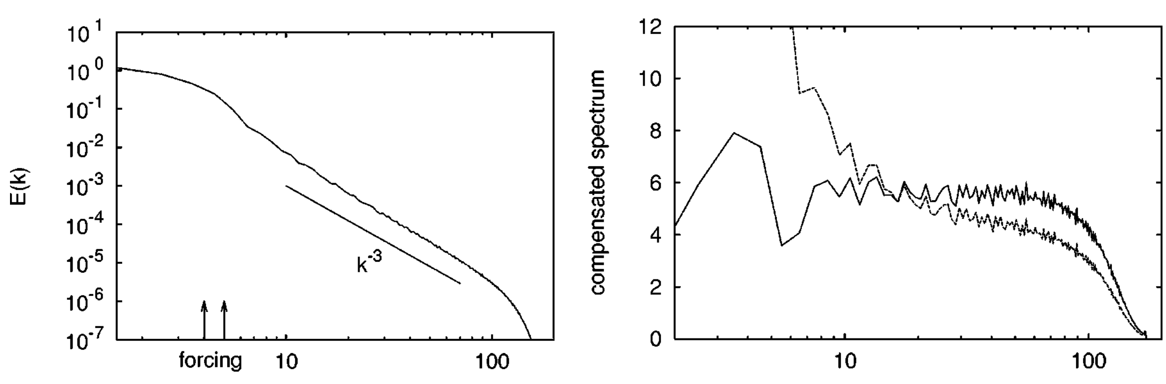

FIG. 2. Energy spectrum (upper panel) and compensated energy spectra (lower panel), where the compensation function is $\eta^{2/3}k^3$ (dashed line) and $\eta^{2/3}k^3[\ln(k/k_F)]^{1/3}$ (solid line).

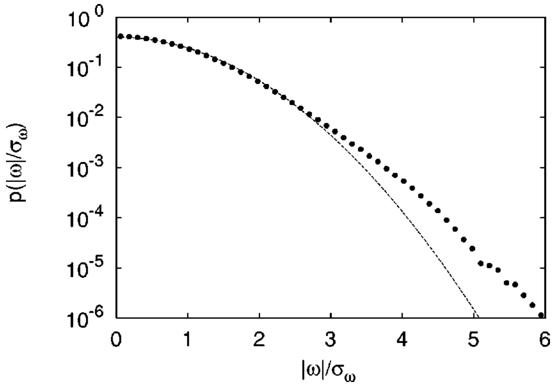

FIG. 3. Time averaged probability distribution function of the modulus of the normalized vorticity. The dashed line indicates a Gaussian distribution with the same variance.

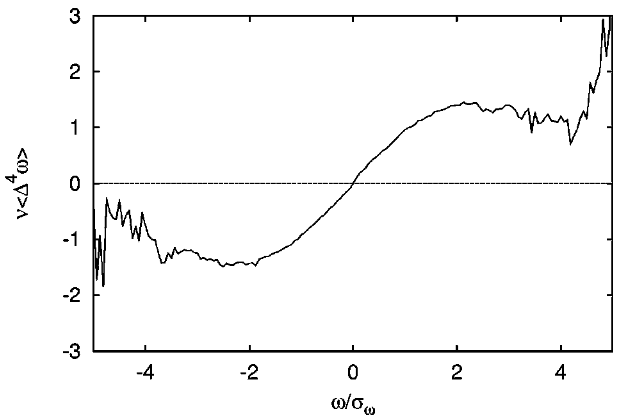

FIG. 4. Conditional average of the viscous term as a function of the vorticity in the point where the iterated Laplacian is calculated.

where forcing and dissipation are both negligible. Those works thus refer to a totally different regime.

In Fig. 3 the probability density function (PDF) of the modulus of the normalized vorticity $|\omega|/\sigma_\omega$ is shown. Here σ_ω is the vorticity rms. The vorticity distribution has tails higher than a Gaussian distribution (shown with the dashed line in the same figure). The large statistics that we have gives confidence up to 6 σ_ω in the tail of the distribution. Note that vorticity PDF found here is similar to the passive scalar probability density function that is known to have a Gaussian core and exponential tails [4,18]. The non-Gaussianity of the vorticity PDF is due to an interplay between advection and viscosity. In the absence of the viscous term, the vorticity ω^* of a fluid particle satisfies a Langevin equation $d\omega^*/dt = f(t)$ in the Lagrangian frame. The single-point vorticity PDF is then Gaussian under Gaussian pumping. When viscous dissipation is taken into account, the evolution equation for the vorticity ω^* of a fluid particle has a term that depends not only on ω^*, but also on the spatial distribution of the vorticity field:

$$d\omega^*/dt = -\mu\omega^* - \nu\nabla^8\omega(\omega^*) + f(t). \tag{5}$$

In a field with spatial and phase correlations, the conditional average $\langle\nabla^8\omega\rangle_{\omega*}$ over all the fluid particles with $\omega=\omega^*$ can be a nonlinear function of ω^* (see Fig. 4 for the present case). The distribution of $\nabla^8\omega(\omega^*)$ around the mean value $\langle\nabla^8\omega\rangle_{\omega*}$ can also be non-Gaussian. The presence of nonlinearity in Eq. (5) is sufficient to get a stationary vorticity PDF that is non-Gaussian. The physical meaning of this nonlinearity and non-Gaussianity is transparent: the spatial structure of the field is such that dissipation on fluid particles with large vorticity is weaker than in a random field with the same variance. The probability of having fluid particles with large vorticity is then larger than for a Gaussian, as seen from the tails of the distribution in Fig. 3.

To conclude, in this study we show the presence of the logarithmic correction in the energy spectrum of the direct cascade of two-dimensional turbulence and observe the non-Gaussianity of the vorticity PDF. A moderate resolution (512×512) has been used to allow a very long integration (more than a thousand eddy turnover times). This was necessary because forced two-dimensional turbulence can vary on long time scales. Our long integration ensures that the system is really in a statistically stationary state and allows for the computation of time averaged spectra that have a higher confidence. In a recent paper Lindborg and Alvelius [9] report simulations with a higher resolution (4096×4096) but on a much shorter time interval (the whole length of the simulation was about 23 eddy turnover times that is too short to allow for a stationary statistics).

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